The Inverse Chirp-Z Transform for VNA Time Domain Processing

Introduction

In an earlier paper, “VNA Time Domain Processing”, a Copper Mountain Technologies Vector Network Analyzer (VNA) was used to perform Time Domain analysis of S-Parameter reflection measurements. Later in that same paper, Time Domain gating was performed using a Python script and an Inverse Discrete Fourier Transform (IDFT) and a Discrete Fourier Transform (DFT) to get back and forth between the time and frequency domain.

The VNA Time Domain conversion had good resolution:

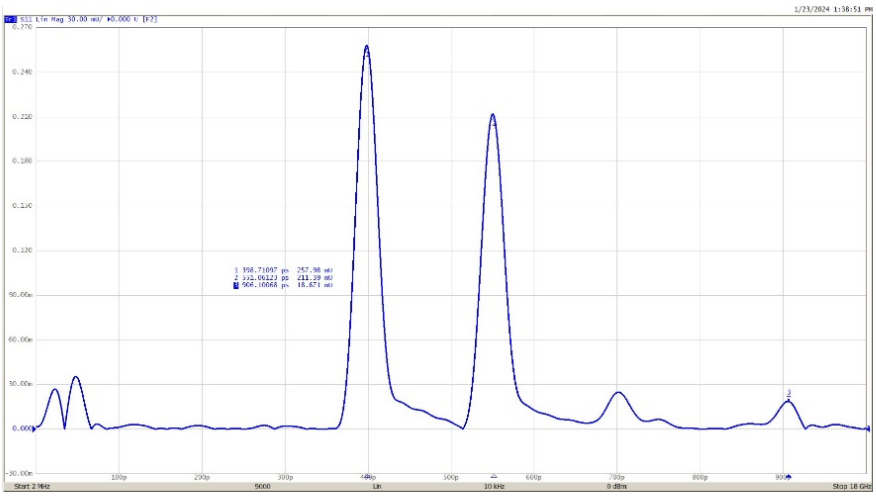

Figure 1 - Time Domain Analysis using the VNA

However, when this same data was converted to Time Domain using an IDFT in Python, the results were not as good:

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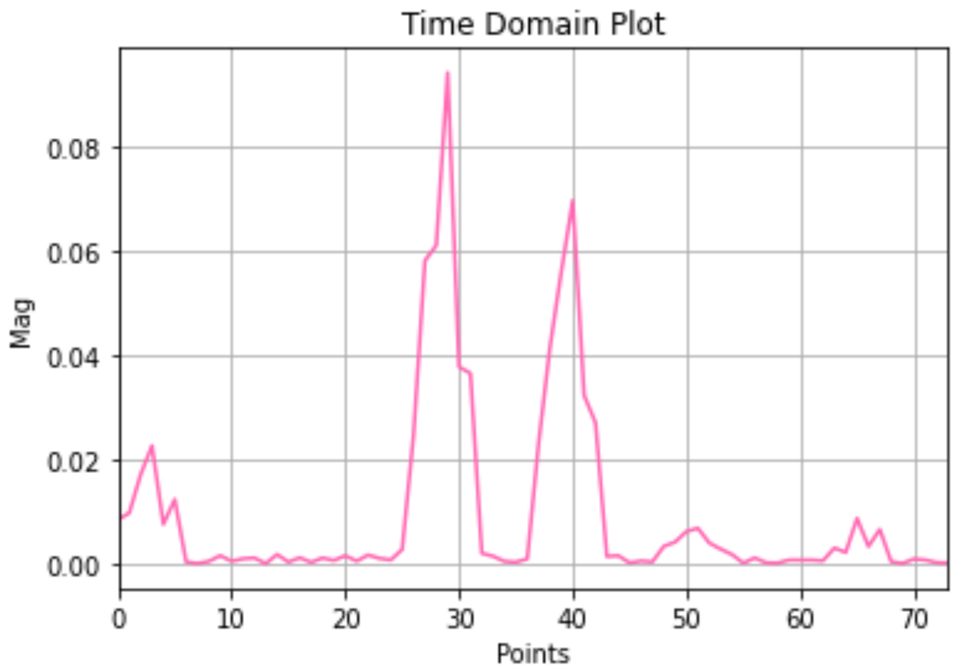

Figure 2 - Time Domain Calculated with IDFT

The result of the IDFT might have thousands of points, but in this case the relevant data is in the first 70 points, so the resolution is terrible. Fortunately, the VNA does not use an IDFT to calculate the Time Domain response, it uses an Inverse Chirp-Z transform which allows for zooming in on selected time intervals with good resolution.

What is the Chirp-Z Transform?

To understand the Chirp-Z Transform (CZT), start with the Fourier Transform, which is defined by:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

For time-valued function f(t). The equation amounts to finding the correlation between the entire range of sinusoids represented by the exponential and the function f(t). After the integral is performed, and the limits applied, the t variable vanishes, leaving only a function of ω.

Alternatively, function f could be a function of distance instead of time and the function f(x) would be integrated over wavenumber “k”:

$$F(k) = \int_{-\infty}^{\infty} f(x)\, e^{-ikx} dx$$

This form is most useful for analyzing waves within bounded physical systems, such as waveguides, resonant cavities and antennas.

The discrete time version of the Fourier Transform is the Discrete Time Fourier Transform (DTFT), or DFT for short. Here, a summation of discrete points is used in place of the integral:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

Where x(n) are the discrete samples in time. The values need not be infinite in extent and might range from index zero to some discrete value. Naturally, Periodicity is assumed for finite data ranges. This summation finds the correlation of the test frequencies in the $e^{-j\omega n}$ term to the data contained in x(n). The use of the exponential form allows for testing any phase of a test frequency ω, since it contains both sine and cosine terms.

The z transform is the more general case of the Discrete Fourier Transform. If $z = e^{j\omega}$, the two transforms are identical:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

In the general case, z can be any complex number with any magnitude or phase.

Why Go to All This Trouble?

We would like to be able to compute the Time Domain response of a reflection sweep as in Figure *1* with good resolution over this narrow time span. A normal IDFT computed on thousands of points of data might give a time range from zero to tens of nanoseconds. But if fixture reflections occur during the first nanosecond, then zooming in by throwing away the extra time data gives the poor result of Figure *2*. The smooth plot of Figure *1* was obtained using the Inverse Chirp-Z Transform (ICZT) as calculated by the VNA. The ICZT allows a Time Domain computation with many points over a specified narrower zoomed in range of time, say, the first nanosecond.

What is the Chirp-Z Transform?

The Chirp-Z Transform[3] is a variation of the Z Transform and is given by:

$$X(k) = \sum_{n=0}^{N-1} x(n)\, A^{-n}\, W^{nk}$$

Where:

$$A = A_0 e^{j2\pi\Theta_0}$$

And:

$$W = W_0 e^{-j2\pi\phi_0}$$

k ranges over M points, $0 \le k \le M-1$ and M need not be equal to N.

Here, the kernel in the right-hand parenthesis describes a complex curve, which starts at radius A_0 and angle Θ_0 and steps in angular increments of Φ_0 and spirals either in or out depending on constant W_0.

The test frequencies of a DFT cover the unit circle equally from 0 to 2π, as in Figure *3*. On the other hand, chirp-Z test frequencies cover an arc, which may or may not encircle the unit circle. Two such arcs are shown in Figure *4*. Both start at magnitude $A_0 = 0.85$ and $\Theta_0 = 20$ degrees, with $\Phi_0 = 8$ degree steps while one arcs outward with $W_0 = 1.05$ and the other inward with $W_0 = 0.99$.

For our purposes, we can set $A_0 = W_0 = 1$ and perform our analysis on the unit circle where we are testing our data with constant amplitude sinusoids just like the DFT. Going off the unit circle allows testing the data for sinusoids, which are exponentially dampened or expanding in amplitude; not a test we need to perform. We can use the capability of the Inverse Chirp-Z to analyze over a small section of the unit circle instead of having to go all the way around from 0 to 2π, effectively zooming in on a particular range of time. We will also choose M = N so that the matrices in the ICZT calculation are square and may be inverted.

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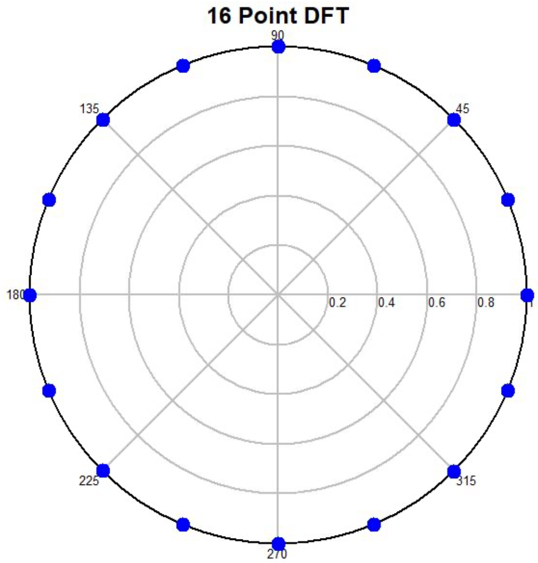

Figure 3 - DFT Test Frequencies

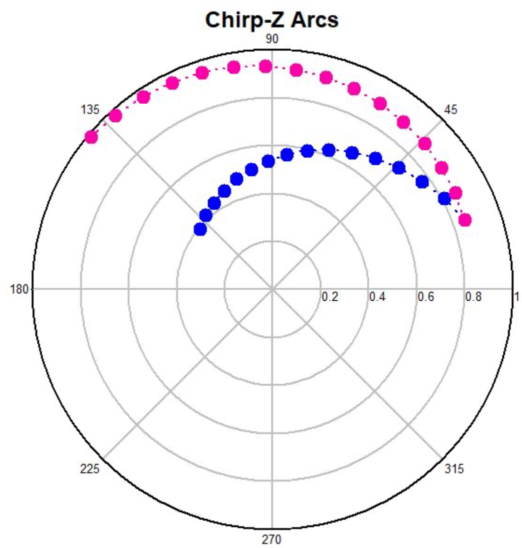

Figure 4 -Chirp Z Test Frequencies

If the Chirp-Z Transform can be performed over a limited range as in Figure *5* on the circumference of the unit circle and its inverse can be bounded similarly, then we have the same capability as the DFT but with the ability to zoom in on a smaller time range.

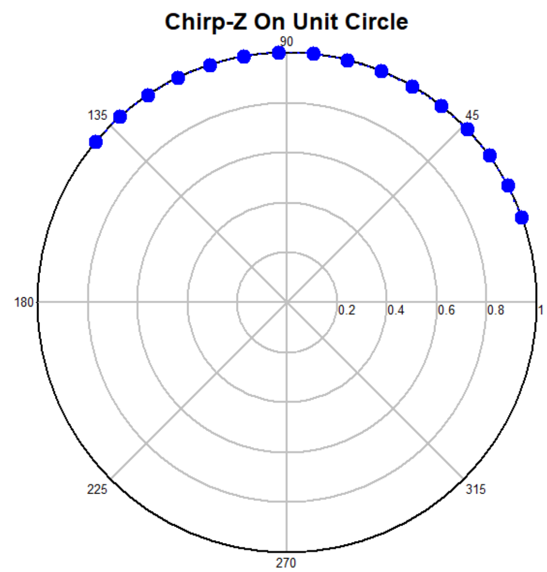

Figure 5 - Chirp Z on Unit Circle

What is the Inverse Chirp-Z Transform?

The Inverse Chirp-Z transform may be used to convert the frequency domain VNA measurements to a specified time domain interval. If the sweep data contains 1,000 equally spaced frequency points with step size dF, the IDFT would also deliver a 1,000-point time domain result from time 0 to 1/dF. If interesting information is contained only in the first 20 points, or in some 20 point interval within the overall time period, the result will have low resolution and look like Figure 2. Using the ICZT algorithm[1,4] one can obtain a 1,000-point time domain result over the smaller time range covered by the first 20 points of the IDFT results, or any other sub-interval.

It should be noted that the accuracy of the Inverse Chirp-Z Time Domain conversion suffers as the interval is made smaller. If the range includes the entire circumference as the DFT, the accuracy will be extremely good, but analysis over a small slice will be poor[2].

Conclusion:

The algorithm for computing the Inverse Chirp-Z transform is complicated and beyond the scope of this paper, but it should be clear that the method allows for analysis of time domain signals over specified ranges from frequency domain VNA measurements. This can be used to analyze localized impairments in transmission lines. Reflections due to adapters or connectors may be singled out of a measurement. Reflections originating at a particular time or distance along a transmission line may be squashed mathematically and, after returning to the frequency domain, the new result may be viewed to determine system performance without this reflection. This is called “Time Domain Gating”.

Copper Mountain Technologies produces a large range of Vector Network Analyzers covering frequencies from 9 kHz to 330 GHz. Premium analytical features such as Time Domain analysis and Time Domain Gating are provided as a standard feature on all VNA models except for the lower cost M series. Please visit us at www.coppermountaintech.com to see the products and to peruse our technical articles, videos and webinars.

References

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